Development of Nonlinear Analysis Tools Based on a Merged IQC/SOS Theory

Final Report

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1 Overview

The connections between dissipation inequalities and integral quadratic constraints (IQCs) was the major thrust explored during the program. In particular, it was shown that existing frequency domain IQC analysis conditions are equivalent (under mild technical assumptions) to a related time-domain dissipation inequality condition. This time-domain approach enables applications of IQCs to analyze the robustness of uncertain time-varying and nonlinear systems. These theoretical techniques can be used to improve the design and robustness of advanced flight control algorithms.

This report documents the research performed as part of the project entitled "Development of Nonlinear Analysis Tools Based on a Merged IQC/SOS Theory". This research is funded by the AFOSR under grant FA9550-12-1-0339. The technical monitor for the program is Dr. Fariba Fahroo. The following subsections summarize the key findings of the research supported on this contract. Details can be found in the publications listed in Section 7.

2 Dissipation Inequalities and Integral Quadratic Constraints

Integral quadratic constraints (IQCs), introduced in [9–11], provide a general framework for robustness analysis. In this framework the system is separated into a feedback connection of a known linear time-invariant (LTI) system and a perturbation whose input-output behavior is described by an IQC. The IQC stability theorem in [9–11] was formulated with frequency domain conditions and was proved using a homotopy method. The remainder of this section briefly describes stability theorems using dissipation inequalities and integral quadratic constraints. The main contribution of the work was to show an equivalence between dissipation theory and IQC approaches [8,21,23]. The benefit of the time-domain dissipation inequality approach is that it can be generalized to cases where the known plant in the feedback interconnection is nonlinear and/or time-varying. For example, dissipation inequality conditions for linear parameter varying systems [24] can be extended to include uncertainty. Details on this work can be found in Reference 2 of the publications listed in Section 7.

2.1 Problem Formulation

Consider the feedback interconnection shown in Figure 1. This interconnection is specified by the following equations:

$$v = Gu + f, \qquad u = \Delta(v) + r \tag{1}$$

where $r \in L_{2e}^m[0,\infty)$ and $f \in L_{2e}^n[0,\infty)$ are exogenous inputs. $\Delta : L_{2e}^n[0,\infty) \to L_{2e}^m[0,\infty)$ is a causal operator with bounded gain. G is a linear time-invariant system:

$$\dot{x}_G = Ax_G + Bu, \qquad y = Cx_G + Du \tag{2}$$

where $x_G \in \mathbb{R}^{n_G}$ is the state of G.

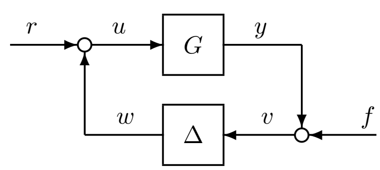

Figure 1: Feedback interconnection

Definition 1 *The interconnection of G and Δ is well-posed if for each $r \in L_{2e}^m[0,\infty)$ and $f \in L_{2e}^n[0,\infty)$ there exist unique $u \in L_{2e}^m[0,\infty)$ and $v \in L_{2e}^n[0,\infty)$ such that the mapping from (r,f) to (u,v) is causal.*

Definition 2 *The interconnection of G and Δ is stable if it is well-posed and if the mapping from (r,f) to (u,v) has finite L_2 gain for all solutions starting from $x_G(0) = 0$.*

2.2 Frequency Domain IQC Stability Condition

Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(n+m)\times(n+m)}$ be a measurable Hermitian-valued function. Two signals $v \in L_2^n[0,\infty)$ and $w \in L_2^m[0,\infty)$ satisfy the IQC defined by the multiplier Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{3}$$

where $\hat{v}(j\omega)$ and $\hat{w}(j\omega)$ are Fourier transforms of v and w. A bounded, causal operator $\Delta : L_{2e}^n[0,\infty) \to L_{2e}^m[0,\infty)$ satisfies the IQC defined by Π if Equation 3 holds for all $v \in L_2^n[0,\infty)$ and $w = \Delta(v)$. The next theorem provides a stability condition for the interconnection of G and Δ.

Theorem 1 ([11]) *Let $G \in \mathbb{RH}_\infty^{n\times m}$ and $\Delta : L_{2e}^n \to L_{2e}^m$ be a bounded causal operator. Assume for all $\tau \in [0,1]$:*

1. *the interconnection of G and $\tau\Delta$ is well-posed.*
2. *$\tau\Delta$ satisfies the IQC defined by Π.*

3. $\exists \epsilon > 0$ *such that*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \quad \forall \omega \in \mathbb{R}. \tag{4}$$

Then the feedback interconnection of G *and* Δ *is stable.*

For rational multipliers, Condition 3 is equivalent to an LMI. Specifically, any $\Pi \in \mathbb{RL}_\infty^{(n+m)\times(n+m)}$ can be factorized as $\Pi = \Psi^\sim M \Psi$ where $M = M^T \in \mathbb{R}^{n_z \times n_z}$ and $\Psi \in \mathbb{RH}_\infty^{n_z \times (n+m)}$. Such factorizations are not unique but can be computed with state-space methods [16]. Denote a state-space realization of Ψ by $(A_\psi, [B_{\psi 1}, B_{\psi 2}], C_\psi, [D_{\psi 1}, D_{\psi 2}])$ where the B_ψ / D_ψ matrices are partitioned compatibly with $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$. A state-space realization for the system $\Psi \left[\begin{smallmatrix} G \\ I \end{smallmatrix}\right]$ is:

$$(\hat{A}, \hat{B}, \hat{C}, \hat{D}) := \left(\begin{bmatrix} A & 0 \\ B_{\psi 1} C & A_\psi \end{bmatrix}, \begin{bmatrix} B \\ B_{\psi 2} + B_{\psi 1} D \end{bmatrix}, \begin{bmatrix} D_{\psi 1} C & C_\psi \end{bmatrix}, D_{\psi 2} + D_{\psi 1} D \right) \tag{5}$$

Finally, the KYP Lemma [14,20] can be applied to demonstrate the equivalence of Condition 3 in Theorem 1 to an LMI condition. This result is stated formally below.

Theorem 2 $\exists \epsilon > 0$ *such that Equation 4 holds if and only if there exists a matrix* $P = P^T$ *such that*

$$\begin{bmatrix} \hat{A}^T P + P \hat{A} & P \hat{B} \\ \hat{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \hat{C}^T \\ \hat{D}^T \end{bmatrix} M \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix} < 0 \tag{6}$$

2.3 Time Domain Dissipation Inequality Stability Condition

An alternative time-domain stability condition can be constructed using IQCs and dissipation theory. Let (Ψ, M) be a factorization of Π. Let signals (v, w) satisfy the IQC in Equation 3 and define $\hat{z}(j\omega) := \Psi(j\omega) \left[\begin{smallmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{smallmatrix}\right]$. Then the IQC can be written as $\int_{-\infty}^{\infty} \hat{z}(j\omega)^* M \hat{z}(j\omega) d\omega \geq 0$. By Parseval's theorem [27], this frequency-domain inequality can be equivalently expressed in the time-domain as:

$$\int_0^\infty z(t)^T M z(t)\, dt \geq 0 \tag{7}$$

where z is the output of the LTI system Ψ:

$$\dot{\psi}(t) = A_\psi \psi(t) + B_{\psi 1} v(t) + B_{\psi 2} w(t), \quad \psi(0) = 0 \tag{8}$$

$$z(t) = C_\psi \psi(t) + D_{\psi 1} v(t) + D_{\psi 2} w(t) \tag{9}$$

Thus Δ satisfies the IQC defined by $\Pi = \Psi^\sim M \Psi$ if and only if the filtered signal $z = \Psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$ satisfies the time domain constraint (Equation 7) for all $v \in L_2^n[0, \infty)$ and $w = \Delta(v)$.

The constraint in Equation 7 holds, in general, only over infinite time. The term hard IQC in [11] refers to the more restrictive property: $\int_0^T z(t)^T M z(t)\ dt \geq 0$ holds $\forall T \geq 0$. In contrast, IQCs for which the time domain constraint need not hold for all finite times are called soft IQCs. This distinction is important because the dissipation theorem below requires the use of hard IQCs. One issue is that the factorization of Π is not unique. Thus the hard/soft property is not inherent to the multiplier Π but instead depends on the factorization (Ψ, M). A more precise definition is now given.

Definition 3 *Let* $\Pi \in \mathbb{RL}_\infty^{(n+m)\times(n+m)}$ *be factorized as* $\Psi^\sim M \Psi$ *where* $M = M^T \in \mathbb{R}^{n_z \times n_z}$ *and* $\Psi \in \mathbb{RH}_\infty^{n_z \times (n+m)}$. *Then* (Ψ, M) *is a hard IQC factorization of* Π *if for any bounded, causal operator* Δ *satisfying the IQC defined by* Π *the following inequality holds*

$$\int_0^T z(t)^T M z(t)\ dt \geq 0 \tag{10}$$

for all $T \geq 0$, $v \in L_{2e}^n[0, \infty)$, $w = \Delta(v)$, *and* $z = \Psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$.

The stability of the feedback system can be analyzed using Figure 2. This feedback interconnection including Ψ is described by $w = \Delta(v)$ and the following extended linear dynamics (omitting the dependence of all signals on time t):

$$\dot{x} = \hat{A}x + \hat{B}w + \hat{B}_2 \left[\begin{smallmatrix} f \\ r \end{smallmatrix}\right] := F(x, w, f, r) \tag{11}$$

$$\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] = \hat{C}_1 x + \hat{D}_{11} w + \hat{D}_{12} \left[\begin{smallmatrix} f \\ r \end{smallmatrix}\right] \tag{12}$$

$$z = \hat{C}x + \hat{D}w + \hat{D}_{22} \left[\begin{smallmatrix} f \\ r \end{smallmatrix}\right] \tag{13}$$

where $x := [x_G^T,\ \psi^T]^T \in \mathbb{R}^{n_G+n_\psi}$ is the extended state. $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are defined in Equation 5. The remaining state matrices are defined as:

$$\hat{B}_2 := \begin{bmatrix} 0 & B \\ B_{\psi 1} & B_{\psi 1} D \end{bmatrix},\ \hat{C}_1 := \left[\begin{smallmatrix} C & 0 \\ 0 & 0 \end{smallmatrix}\right] \tag{14}$$

$$\hat{D}_{11} := \left[\begin{smallmatrix} D \\ I \end{smallmatrix}\right],\ \hat{D}_{12} := \left[\begin{smallmatrix} I & D \\ 0 & I \end{smallmatrix}\right],\ \hat{D}_{22} := \left[\begin{smallmatrix} D_{\psi 1} & D_{\psi 1} D \end{smallmatrix}\right] \tag{15}$$

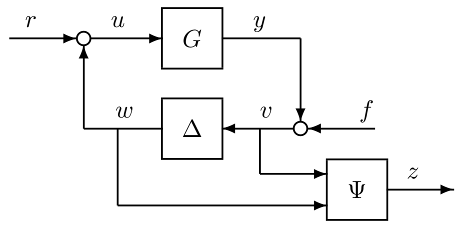

Figure 2: Analysis Interconnection Structure

The next theorem provides a stability condition using IQCs and a standard dissipation argument.

Theorem 3 *Let $G \in \mathbb{RH}_\infty^{n \times m}$ and $\Delta : L_{2e}^n \to L_{2e}^m$ be a bounded causal operator. Assume that:*

1. *the interconnection of G and Δ is well-posed.*
2. *Δ satisfies the IQC defined by Π and (Ψ, M) is a hard factorization of Π.*
3. *there exists $P \geq 0$ and a scalar $\gamma > 0$ such that $V(x) := x^T P x$ satisfies*

$$z^T M z + \nabla V \cdot F(x, w, f, r) < \gamma \left[\begin{smallmatrix} r \\ f \end{smallmatrix}\right]^T \left[\begin{smallmatrix} r \\ f \end{smallmatrix}\right] - \frac{1}{\gamma} \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]^T \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \tag{16}$$

for all nontrivial $(x, w, r, f) \in \mathbb{R}^{n_G+n_\psi} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$ where u, v, z are defined by Equations 12 and 13.

Then the feedback interconnection of G and Δ is stable.

Equation 16 is an algebraic inequality on the variables (x, w, f, r). This constraint, when evaluated along solutions of the extended system, represents the differential form for a dissipation inequality satisfied by the extended system. The next lemma shows that the dissipation inequality in Equation 16 is also equivalent to the KYP LMI.

Lemma 1 *There exists $P \geq 0$ satisfying the dissipation inequality (Equation 16) for some $\gamma > 0$ if and only if there exists $P \geq 0$ satisfying the KYP LMI (Equation 6).*

2.4 Main Result

The previous section summarizes two IQC stability theorems. Theorem 1 involves a frequency domain condition with a multiplier Π. Theorem 3 involves a dissipation inequality with a multiplier (Ψ, M). The multipliers are connected by a non-unique factorization $\Pi = \Psi^{\sim} M \Psi$. Theorems 1 and 3 are clearly related as the Condition 3 in each theorem is equivalent to the same KYP LMI. Two important properties are required for the dissipation inequality approach:

1. (Ψ, M) must be a "hard" factorization to ensure the time-domain constraint holds over all finite intervals.

2. The solution to the KYP LMI must satisfy $P \geq 0$. This is not required for the frequency domain test.

The main result is: For a class of multipliers, Π has a factorization (Ψ, M) that is both "hard" and such that any feasible solution of the KYP LMI satisfies $P \geq 0$.

2.4.1 Condition for Hard Factorization

Define the following cost functional J on $v \in L_2^n[0,\infty)$, $w \in L_2^m[0,\infty)$, and $\psi_0 \in \mathbb{R}^{n_\psi}$:

$$\begin{aligned} J(v, w, \psi_0) := \int_0^\infty & z(t)^T M z(t)\, dt \\ & \text{subject to:} \\ & \dot{\psi}(t) = A_\psi \psi(t) + B_{\psi 1} v(t) + B_{\psi 2} w(t),\ \psi(0) = \psi_0 \\ & z(t) = C_\psi \psi(t) + D_{\psi 1} v(t) + D_{\psi 2} w(t) \end{aligned} \tag{17}$$

Also define the upper value $\bar{J}$ as

$$\bar{J}(\psi_0) := \inf_{v \in L_2^n[0,\infty)} \sup_{w \in L_2^m[0,\infty)} J(v, w, \psi_0) \tag{18}$$

Lemma 2 *Let $\Pi \in \mathbb{RL}_\infty^{(n+m)\times(n+m)}$ be a multiplier and (Ψ, M) any factorization of Π with Ψ stable. Assume $\Delta : L_{2e}^n[0,\infty) \to L_{2e}^m[0,\infty)$ is a casual, bounded operator that satisfies the IQC defined by Π. Then for all $T \geq 0$, $v \in L_{2e}^m[0,\infty)$ and $w = \Delta(v)$, the output of Ψ satisfies:*

$$\int_0^T z(t)^T M z(t)\, dt \geq -\bar{J}(\psi_T) \tag{19}$$

where ψ_T denotes the state of Ψ at time T when driven by inputs (v, w) with initial condition $\psi(0) = 0$.

2.4.2 Condition for Positive Semidefinite KYP Solution

Define the lower value $\underline{J}$ as

$$\underline{J}(\psi_0) := \sup_{w \in L_2^m[0,\infty)} \inf_{v \in L_2^n[0,\infty)} J(v, w, \psi_0) \tag{20}$$

Lemma 3 *Let $\Pi \in \mathbb{RL}_\infty^{(n+m)\times(n+m)}$ be a multiplier and (Ψ, M) any factorization of Π with Ψ stable. Given $G \in \mathbb{RH}_\infty^{n\times m}$, assume the corresponding KYP LMI (Equation 6) is feasible with state matrices $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ defined in Equation 5. Let $P = P^T$ denote a solution to the KYP LMI. Then $V(x_0) := x_0^T P x_0 \geq \underline{J}(\psi_0)$ for all $x_0 := [\, x_{G,0}^T, \psi_0^T \,]^T \in \mathbb{R}^{n_G + n_\psi}$.*

2.4.3 Dissipation Inequalities with J-Spectral Factorizations

By Lemma 2, (Ψ, M) is a hard factorization if $\bar{J}(\psi) \leq 0 \ \forall \psi$. By Lemma 3, all KYP LMI solutions satisfy $P \geq 0$ if $\underline{J}(\psi) \geq 0 \ \forall \psi$. Moreover, weak duality implies that the lower and upper values satisfy $\underline{J}(\psi) \leq \bar{J}(\psi)$. Hence a factorization $\Pi = \Psi^{\sim} M \Psi$ that is both "hard" and ensures $P \geq 0$ for all KYP LMI solutions must have $0 \leq \underline{J}(\psi) \leq \bar{J}(\psi) \leq 0$. In other words, for such a factorization the lower and upper values must satisfy $\underline{J}(\psi) = \bar{J}(\psi) = 0$. The following special factorization plays a key role in the main result below.

Definition 4 *(Ψ, M) is called a $J_{n,m}$-spectral factor of $\Pi = \Pi^{\sim} \in \mathbb{RL}_{\infty}^{(n+m)\times(n+m)}$ if $\Pi = \Psi^{\sim} M \Psi$, $M = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix}$, and $\Psi, \Psi^{-1} \in \mathbb{RH}_{\infty}^{(n+m)\times(n+m)}$.*

Reference 2 in the publications funded by this project (Section 7) provides sufficient conditions for the existence of a J-spectral factorization. These conditions are used in the main result stated below.

Theorem 4 *Let $\Pi = \Pi^{\sim} \in \mathbb{RL}_{\infty}^{(n+m)\times(n+m)}$ and partition as $\begin{bmatrix} \Pi_{11} & \Pi_{21}^{\sim} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$ where $\Pi_{11} \in \mathbb{RL}_{\infty}^{n\times n}$ and $\Pi_{22} \in \mathbb{RL}_{\infty}^{m\times m}$. If $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0 \ \forall \omega \in \mathbb{R} \cup \{\infty\}$, then*

1. *Π has a $J_{n,m}$-spectral factorization (Ψ, M).*
2. *The $J_{n,m}$-spectral factorization (Ψ, M) is a hard factorization of Π.*
3. *For $G \in \mathbb{RH}_{\infty}^{n\times m}$, let $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ denote the state-space realization of $\Psi \begin{bmatrix} G \\ I \end{bmatrix}$ in Equation 5. All solutions $P = P^T$ to the KYP LMI (Equation 6) satisfy $P \geq 0$.*

Factorization conditions in [4, 5] connect classical passivity multipliers and their IQC counterparts. Theorem 4 provides a connection between classical passivity multipliers and dissipation theory. Specifically, let H be a classical passivity multiplier proving stability for the interconnection of G and a finite-gain system Δ. It follows by a simple perturbation argument, e.g. as in [4], that stability can be demonstrated with the (frequency-domain) IQC test using $\Pi = \begin{bmatrix} \epsilon I & H^* \\ H & \frac{-\epsilon}{\|\Delta\|^2} I \end{bmatrix}$. The conditions in Theorem 4 hold for this multiplier and thus a J-spectral factorization of Π exists. Moreover, there is a dissipation inequality that proves stability of the feedback interconnection. In other words, if stability can be demonstrated by a classical passivity multiplier then it can also be demonstrated via a dissipation inequality.

3 Robustness Analysis for Linear Parameter Varying Systems

The main result in the previous section connects dissipation inequalities and integral quadratic constraints. As mentioned previously, this enables new applications of IQCs to analyze the robustness of time-varying and nonlinear systems. This section considers the robustness of uncertain linear parameter varying (LPV) systems. Details on the results contained in this section can be found in Reference 3 of the publications generated by this research (Section 7).

The uncertain system is described by the feedback interconnection of an LPV system G and an uncertainty Δ. This feedback interconnection with Δ wrapped around the top of G is denoted $F_u(G, \Delta)$. The LPV system G is a linear system whose state space matrices depend on a time-varying parameter vector $\rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho}$ as follows:

$$\begin{aligned} \dot{x}_G(t) &= A_G(\rho(t))\, x_G(t) + B_G(\rho(t)) \begin{bmatrix} w(t) \\ d(t) \end{bmatrix} \\ \begin{bmatrix} v(t) \\ e(t) \end{bmatrix} &= C_G(\rho(t))\, x_G(t) + D_G(\rho(t)) \begin{bmatrix} w(t) \\ d(t) \end{bmatrix} \end{aligned} \tag{21}$$

where $x_G \in \mathbb{R}^{n_G}$ is the state, $w \in \mathbb{R}^{n_w}$ and $d \in \mathbb{R}^{n_d}$ are inputs, and $v \in \mathbb{R}^{n_v}$ and $e \in \mathbb{R}^{n_e}$ are outputs. The state matrices of G have dimensions compatible with these signals, e.g. $A_G(\rho) \in \mathbb{R}^{n_G \times n_G}$. In addition, the state matrices are assumed to be continuous functions of ρ. The state matrices at time t depend on the parameter vector at time t. Hence, LPV systems represent a special class of time-varying systems. The explicit dependence on t is occasionally suppressed to shorten the notation. Moreover, it is important to emphasize that the state matrices are allowed to have an arbitrary dependence on the parameters. This is called a "gridded" LPV system and is more general than "LFT" LPV systems whose state matrices are restricted to have a rational dependence on the parameters [1, 12, 15].

The parameter ρ is assumed to be a continuously differentiable function of time and admissible trajectories are restricted to a known compact set $\mathcal{P} \subset \mathbb{R}^{n_\rho}$. In addition, the parameter rates of variation $\dot{\rho} : \mathbb{R}^+ \to \dot{\mathcal{P}}$ are assumed to lie within a hyperrectangle $\dot{\mathcal{P}} := \{q \in \mathbb{R}^{n_\rho} | \ \underline{\nu}_i \leq q_i \leq \bar{\nu}_i, \ i = 1, \ldots, n_\rho\}$. The set of admissible trajectories is defined as

$$\mathcal{T} := \left\{ \rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho} \ : \ \rho \in \mathcal{C}^1, \ \rho(t) \in \mathcal{P} \text{ and } \dot{\rho}(t) \in \dot{\mathcal{P}} \ \forall t \geq 0 \right\} \tag{22}$$

The parameter trajectory is said to be rate unbounded if $\dot{\mathcal{P}} = \mathbb{R}^{n_\rho}$.

Throughout the section it is assumed that the uncertain system has a form of nominal stability. Specifically, G is assumed to be parametrically-dependent stable as defined in [25].

Definition 5 *G is parametrically-dependent stable if there is a continuously differentiable function $P : \mathbb{R}^{n_\rho} \to \mathcal{S}^{n_G \times n_G}$ such that $P(p) \geq 0$ and*

$$A_G(p)^T P(p) + P(p) A_G(p) + \sum_{i=1}^{n_\rho} \frac{\partial P}{\partial p_i} q_i < 0 \tag{23}$$

hold for all $p \in \mathcal{P}$ and all $q \in \dot{\mathcal{P}}$.

As discussed in [25], parametric-stability implies G has a strong form of robustness. In particular, the state $x_G(t)$ of the autonomous response ($w = 0$, $d = 0$) decays exponentially to zero for any initial condition $x_G(0) \in \mathbb{R}^{n_G}$ and allowable trajectory $\rho \in \mathcal{T}$ (Lemma 3.2.2 of [25]). Moreover, the state $x_G(t)$ of the forced response decays asympotically to zero for any initial condition $x_G(0) \in \mathbb{R}^{n_G}$, allowable trajectory $\rho \in \mathcal{T}$, and inputs $w, d \in L_2$ (Lemma 3.3.2 of [24]). The parameter-dependent Lyapunov function $V(x_G, \rho) := x_G^T P(\rho) x_G$ plays a key role in the proof of these results. To shorten the notation, a differential operator $\partial P : \mathcal{P} \times \dot{\mathcal{P}} \to \mathbb{R}^{n_x}$ is introduced as in [17]. ∂P is defined as $\partial P(p, q) := \sum_{i=1}^{n_\rho} \frac{\partial P}{\partial \rho_i}(p) \, q_i$. This simplifies the expression of Lyapunov-type inequalities similar to Equation 23.

The uncertainty $\Delta : L_{2e}^{n_v}[0, \infty) \to L_{2e}^{n_w}[0, \infty)$ is a bounded, causal operator. The notation $\mathbf{\Delta}$ is used to denote the set of bounded, causal uncertainties Δ. The input/output behavior of the uncertain set is bounded using quadratic constraints as described further in the next section. At this point it is sufficient to state that Δ can have block-structure as is standard in robust control modeling [27]. Δ can include blocks that are hard nonlinearities (e.g. saturations) and infinite dimensional operators (e.g. time delays) in addition to true system uncertainties. The term uncertainty is used for simplicity when referring to the perturbation Δ.

The objective of this section is to assess the robustness of the uncertain system $F_u(G, \Delta)$. For a given $\Delta \in \mathbf{\Delta}$, the induced L_2 gain from d to e is defined as:

$$\|F_u(G, \Delta)\| := \sup_{\substack{0 \neq d \in L_2^{n_d}[0,\infty) \\ \rho \in \mathcal{T}, \ x_G(0)=0}} \frac{\|e\|_2}{\|d\|_2} \tag{24}$$

Two forms of robustness are considered. First, the worst-case induced L_2 gain from input d to the output e is defined as

$$\sup_{\Delta \in \mathbf{\Delta}} \|F_u(G_\rho, \Delta)\|. \tag{25}$$

This is the worst-case gain over all uncertainties $\Delta \in \mathbf{\Delta}$ and admissible trajectories $\rho \in \mathcal{T}$. Second, the system has robust asymptotic stability if $x_G(t) \to 0$ for any initial condition $x_G(0) \in \mathbb{R}^{n_G}$, allowable trajectory $\rho \in \mathcal{T}$, disturbance $d \in L_2$ and uncertainty $\Delta \in \mathbf{\Delta}$.

The main result (Theorem 5 below) provides a sufficient condition for when an uncertain LPV system has both robust asymptotic stability and bounded worst-case gain. The results is based on the interconnection of the nominal LPV system G and the filter Ψ of the IQC factorization, similar to Figure 2. The dynamics of this interconnection are described by $w = \Delta(v)$ and

$$\begin{aligned} \dot{x} &= A(\rho)x + B_1(\rho)w + B_2(\rho)d \\ z &= C_1(\rho)x + D_{11}(\rho)w + D_{12}(\rho)d \\ e &= C_2(\rho)x + D_{21}(\rho)w + D_{22}(\rho)d, \end{aligned} \tag{26}$$

with $x := \begin{bmatrix} x_G \\ \psi \end{bmatrix} \in \mathbb{R}^{n_G+n_\psi}$ is the extended state. Removing the uncertainty Δ from the analysis interconnection, w can be viewed as an external signal subject to the constraint $\int_0^T z^T(t)Mz(t)\,dt \geq 0$.

Theorem 5 *Let G be a parametrically stable LPV system defined by eq.* (21) *and $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ be a bounded, causal operator such that $F_u(G,\Delta)$ is well-posed. Assume Δ satisfies the IQC parameterized by $\Pi(\lambda) = \Psi^\sim M(\lambda)\Psi$ with Ψ stable. If*

1. *The combined multiplier, partitioned as $\Pi(\lambda) = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\sim & \Pi_{22} \end{bmatrix}$, satisfies $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0$ $\forall \omega \in \mathbb{R} \cup \{\infty\}$ where Π_{11} is $n_v \times n_v$ and Π_{22} is $n_w \times n_w$.*

2. *There exists a continuously differentiable $P : \mathcal{P} \to \mathcal{S}^{n_x \times n_x}$, and a scalar $\gamma > 0$ such that*

$$\begin{bmatrix} A^TP+PA+\partial P & PB_1 & PB_2 \\ B_1^TP & 0 & 0 \\ B_2^TP & 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_2^T \\ D_{21}^T \\ D_{22}^T \end{bmatrix} \begin{bmatrix} C_2^T \\ D_{21}^T \\ D_{22}^T \end{bmatrix}^T + \begin{bmatrix} C_1^T \\ D_{11}^T \\ D_{12}^T \end{bmatrix} M(\lambda) \begin{bmatrix} C_1^T \\ D_{11}^T \\ D_{12}^T \end{bmatrix}^T < 0 \tag{27}$$

   hold for all $p \in \mathcal{P}$ and all $q \in \dot{\mathcal{P}}$.

Then

a) *For any $x(0) \in \mathbb{R}^{n_G+n_\psi}$ and $d \in L_2$, $\lim_{T\to\infty} x(T) = 0$*

b) *$\|F_u(G,\Delta)\| \leq \gamma$*

The implementation of Theorem 5 involves some numerical issues. These are briefly described here. If the IQC is parameterized such that $M(\lambda)$ is an affine function of λ then theorem 5 involves parameter dependent LMI conditions in the variables $P(\rho)$ and λ. λ needs to satisfy condition 1 in theorem 5, i.e. $\Pi_{11} > 0$ and $\Pi_{22} < 0$. These are infinite dimensional (one LMI for each $(p,q) \in \mathcal{P} \times \dot{\mathcal{P}}$) and they are typically approximated with finite-dimensional LMIs evaluated on a grid of parameter values. Additionally, the main decision variable is the function $P(\rho)$ which must be restricted to a finite dimensional subspace. A common practice [2, 26] is to restrict $P(\rho)$ to be a linear combination of user-specified basis functions. The analysis can then be performed as a finite-dimensional SDP [3], e.g. minimizing γ subject to the approximate finite-dimensional LMI conditions. This paper focused on gridded LPV systems whose state matrices have an arbitrary dependence on the parameter. If the LPV system has a rational dependence on ρ then finite dimensional LMI conditions can be derived (with no gridding) using the techniques in [1, 12].

4 Nonlinear Robustness Analysis

The main result in Section 2 connects dissipation inequalities and integral quadratic constraints. As mentioned previously, this enables new applications of IQCs to analyze the robustness of time-varying and nonlinear systems. This section considers the analysis of nonlinear systems. Details on the results contained in this section can be found in Reference 1 of the publications generated by this research (Section 7).

Consider a nonlinear system governed by differential equations of the follwoing form

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))w(t), \\ z(t) &= h(x(t)), \end{aligned} \tag{28}$$

where $t \in \mathbb{R}$, $x(0) = x_0 \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^p$, $w(t) \in \mathbb{R}^m$. The functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be Lipschitz continuous, or locally Lipschitz continuous, depending on the situation. If f and g are not Lipschitz continuous (as in the case of polynomial f and g, for example), then the differential equation may exhibit finite escape times in the presence of bounded inputs and/or initial conditions.

The goal of this research is quantitative, local analysis of nonlinear dynamical systems. By "quantitative" we mean algorithms and sufficient conditions which lead to concrete guarantees about a particular system's response. By "local" we refer to guarantees about the reachability and/or system gain which are predicated on assumptions concerning the magnitude of initial conditions and input signals. We extensively use the basic, fundamental ideas from dissipative systems theory [22], [7], barrier functions and reachability [19], [13], and nonlinear optimal control [18], [6]. Specifically, we employ inequalities involving the Lie derivative of a scalar function, the *storage* function, that hold throughout regions of the state and input space, which when integrated over trajectories of the system, give certificates of input/output properties of the system. The necessity of the existence of such storage functions to prove input/output properties, which leads to the most elegant results of the above mentioned works, is actually not used in this paper. Our computational approach is based on polynomial storage functions of fixed degree which can be viewed as extensions of known linear matrix inequality conditions to compute reachable sets and input/output gains for linear systems [3].

The contributions are as follows: a dissipation inequality formulation of local reachability and dissipativeness for uncertain systems that are not nominally globally stable are derived; refinements on the reachability and L_2 gain conditions that can be used to efficiently compute improved quantitative performance bounds; sum-of-squares (SOS) characterizations of the required set containment conditions in the dissipation inequalities; proof of guaranteed feasability of the SOS conditions for systems with stable linearizations; development of a scheme to find feasible solutions to the bilinear SOS conditions, and improve the objective through a specific iteration scheme; and a collection of illustrative and realistic examples illustrating the methods.

One result on reachability is provided to demonstrate the basic approach. Details on the remaining results can be found in Reference 1 of the publications generated by this research (Section 7). Specifically, we establish conditions which guarantee invariance of certain sets under L_2 and pointwise-in-time (L_∞-like) constraints on w. These are subsequently referred to as "reachability" results, since the conclusions yield outer bounds on the set of reachable states. In that vein, w is interpreted as a disturbance, whose worst-case effect on the state x is being quantified. We obtain bounds on x that are tightly linked with the assumed bounds on w and x_0, and specifically allow for systems which are not well-defined on all input signals (finite escape times).

A known set $\mathcal{W} \subseteq \mathbb{R}^m$ is used to express any L_∞-like, pointwise-in-time bound on the signal w, namely $w(t) \in \mathcal{W}$ for all t. Setting $\mathcal{W} = \mathbb{R}^m$ is equivalent to the absence of known, pointwise-in-time bounds on w.

Theorem 6 *Suppose $\mathcal{W} \subseteq \mathbb{R}^m$. Assume that f and g in (28) are Lipschitz continuous on $\mathbb{R}^n$. Suppose $\tau > 0$, and a differentiable $Q : \mathbb{R}^n \to \mathbb{R}$ satisfies $Q(0) < \tau^2$ and*

$$\Omega^{cc,0}_{Q,\tau^2} \times \mathcal{W} \subseteq \left\{(x, w) \in \mathbb{R}^n \times \mathbb{R}^m : \nabla Q(x) \cdot [f(x) + g(x)w] \leq w^T w\right\}. \tag{29}$$

Consider $x_0 \in \Omega^{cc,0}_{Q,\tau^2}$ with $Q(x_0) < \tau^2$ and $w \in L_2^m$ with $w(t) \in \mathcal{W}$ for all t. If $\|w\|_2^2 < \tau^2 - Q(x_0)$, the solution to (28) with $x(0) = x_0$ satisfies $Q(x(t)) < \tau^2$ for all t, and hence $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all t.

Without loss of generality, Q in Theorem 6 can be taken to be zero at $x = 0$. For instance, define $\tilde{Q}(x) := Q(x) - Q(0)$ and $\tilde{\tau}^2 := \tau^2 - Q(0)$. The conditions of Theorem 6 hold with $\tilde{Q}$ replacing Q, and the same norm bound (i.e. reachable set) is obtained. Computational approaches based on the S-procedure and sum-of-squares are described further in the paper.

5 IQC Analysis for Certifying Exponential Convergence

The standard stability result for interconnections involving nonlinearities described by IQCs [9–11] provides a frequency-domain test for certifying BIBO stability of the associated interconnected system. Since the frequency-domain test is difficult to verify computationally, a standard approach is to use the Kalman-Yakubovich-Popov (KYP) lemma to obtain an equivalent linear matrix inequality (LMI) that can then be checked using a conventional convex programming software package.

The approach outlined above is useful in verifying the robust stability of an interconnected system containing components that are nonlinear, uncertain, poorly modeled, or otherwise problematic. With minor modifications, the same result can be adapted for use in certifying L_2 gain bounds or passivity. Unfortunately, the approach cannot prove **exponential** stability. To contrast both notions of stability, we have (in discrete time):

$$\begin{aligned} L_2 \text{ stability implies:} \quad & \sum_{k=0}^{\infty} \|x_k\|^2 < \gamma^2 \qquad \text{for some } \gamma, \text{ while} \\ \text{exponential stability implies:} \quad & \|x_k\| < c\lambda^k \qquad \text{for some } c > 0 \text{ and } \lambda \in (0,1) \end{aligned}$$

Note that in the L_2 case, the norm is invariant under rearrangement. Thus, two signals with the same γ both eventually go to zero, but their transient behaviors may be completely different. In the exponential case, however, two signals with the same (c, λ) have the same decaying exponential envelope. Therefore, exponential stability is much stronger than its L_2 counterpart, and it is useful to be able to certify it.

The standard IQC result can only certify L_2 stability. The authors of [Megretski and Rantzer '97] address this shortcoming by pointing out that BIBO stability often implies exponential stability in many cases of practical interest. So by certifying an L_2 gain bound, we often get exponential stability for free. However, in such cases, one can only prove the *existence* of some (c, λ). One could not, for example, optimize to find the certificate with the fastest possible exponential rate (i.e. minimizing λ).

Under the support of AFOSR, we developed an IQC-based method for certifying exponential convergence that also allows one to optimize over λ. We also provide nontrivial instances for which our exponential rate bounds are tight.

To illustrate our approach, it is useful to first consider a simple case. Consider a discrete linear time-invariant (LTI) plant G with state-space realization (A, B, C, D). Suppose G is connected in feedback with a passive nonlinearity Δ. A sufficient condition for BIBO stability is that there exists a positive definite matrix $P \succ 0$ and a scalar $\lambda \geq 0$ satisfying the linear matrix inequality (LMI)

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & C^T \\ C & D+D^T \end{bmatrix} \prec 0 \tag{30}$$

If we define $V(x) := x^T P x$, then (30) implies that V decreases along trajectories: $V(x_{k+1}) \le V(x_k)$ for all k. BIBO stability then follows from positivity and boundedness of V. But observe that when (30) holds, we may replace the right-hand side by $-\varepsilon P$ for some $\varepsilon > 0$ sufficiently small. We then conclude that $V(x_{k+1}) \le (1-\varepsilon)V(x_k)$ for all k and exponential stability follows. We can then maximize ε subject to feasibility of (30) to further improve the rate bound. A simple way of carrying out this maximization is to perform a bisection search on ε, since (30) is a convex program for every fixed ε.

The simple trick shown above works because passivity is a very special case of an IQC for which there are no dynamics involved. In other words, passivity can be verified in a pointwise fashion by checking that $u_k^T y_k \ge 0$ for all k. Unfortunately, the trick shown above does not work in the general dynamic IQC setting due to the different role played by P in the associated LMI. The LMI used in IQC theory comes from the KYP lemma and although it is structurally similar to (30), P is not positive definite in general and V may not decrease along trajectories.

Our key insight is that with a suitable modification to both the LMI *and* the IQC definition, we obtain a condition that can certify exponential stability for a given rate λ. The resulting optimization problem is convex for any λ, so we can optimize over λ by performing a bisection search. We show that our modified IQC approach finds tight exponential bounds for a simple yet nontrivial example: a third-order plant in feedback with a saturating nonlinearity. Additional details on the results contained in this section can be found in Reference 4 of the publications generated by this research (Section 7).

6 Analyzing Optimization Algorithms

Electromechanical systems often contain embedded optimization algorithms. One example is modern applications of model-predictive control. Other examples include machine learning and computer vision, where large quantities of data must be processed in real-time on a small platform such as a robot or UAV. In all such cases, the optimization algorithms are sequential in nature, and are carried out either for a fixed number of steps, or until a desired error tolerance is reached.

Sequential algorithms can be interpreted as uncertain dynamical systems, where the uncertainty is due to the function being optimized, and the dynamics are due to the choice of sequential algorithm. As a simple example, consider the *heavy ball* method applied to some unknown differentiable function $f : \mathbb{R}^n \to \mathbb{R}$.

$$x_{k+1} = x_k - \alpha \nabla f(x_k) + \beta(x_k - x_{k-1}) \tag{31}$$

Here, α and β are the *stepsize* and *momentum* parameters, respectively. By including additional definitions for p_k, u_k, y_k, we may rewrite (31) as follows:

$$\begin{bmatrix} x_{k+1} \\ p_{k+1} \end{bmatrix} = \begin{bmatrix} 1+\beta & -\beta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ p_k \end{bmatrix} + \begin{bmatrix} -\alpha \\ 0 \end{bmatrix} u_k \tag{32a}$$

$$y_k = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_k \\ p_k \end{bmatrix} \tag{32b}$$

$$u_k = \nabla f(y_k) \tag{32c}$$

If we let G be the linear time-invariant system $G : u \mapsto y$ described by (32a)–(32b) and we let ∇f be the static map $y \mapsto u$ described by (32c), then we can represent (31) as the block diagram of Figure 3.

It turns out that many different kinds of iterative algorithms may be abstracted in the form of Figure 3. These include gradient-based schemes such as gradient descent and its accelerated variants such as the heavy ball method and the fast gradient method (also known as Nesterov's accelerated method). More complicated examples include projected variants for use in constrained optimization, proximal algorithms, and operator-splitting methods such as the alternating direction method of multipliers (ADMM).

A common way of evaluating the performance of an iterative optimization algorithm is to bound its worst-case convergence rate. When viewed as a dynamical system, this amounts to proving exponential

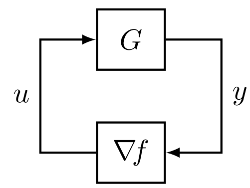

Figure 3: Block diagram representing the uncertain dynamics of the heavy ball method (31). G represents the LTI dynamics of the iterative algorithm while ∇f is the gradient of the function being optimized.

stability and computing the smallest associated decay rate $\lambda \in (0, 1)$. Therefore, the method described in Section 5 is perfectly suited for this analysis.

When we carry out the analysis, we recover the best possible rates for a multitude of algorithms as found in the existing literature on optimization algorithms. Moreover, the same proof works for all the algorithms. In a further study, we used the same IQC-based approach to analyze the ADMM algorithm specifically. We found provably tight bounds on its performance as well as a principled approach to hyperparameter selection. The benefits of our analysis are the inherent simplicity of the approach and the fact that the associated LMIs are very small and can be solved in milliseconds using generic solvers on cheap hardware. Additional details on the results contained in this section can be found in References 5 and 6 of the publications generated by this research (Section 7).

7 Publications

The following is a list of publications that have either been partially supported or fully supported during the this AFOSR project.

1. E. Summers, A. Chakraborty, W. Tan, U. Topcu, P. Seiler, G. Balas, and A. Packard, "Quantitative local L_2-gain and Reachability analysis for nonlinear systems," Int. Journal of Robust and Nonlinear Control, vol. 23, p. 1115-1135, 2013.

2. P. Seiler, "Stability Analysis with Dissipation Inequalities and Integral Quadratic Constraints," to appear in the IEEE Transactions on Automatic Control, 2015.

3. H. Pfifer and P. Seiler, "Less Conservative Robustness Analysis of Linear Parameter Varying Systems Using Integral Quadratic Constraints," submitted to the International Journal of Robust and Nonlinear Control, 2015.

4. R. Boczar, L. Lessard, and B. Recht, "Exponential convergence bounds using integral quadratic constraints," submitted to the 2015 IEEE Conference on Decision and Control.

5. R. Nishihara, L. Lessard, B. Recht, A. Packard, and M. I. Jordan, "A general analysis of the convergence of ADMM," to appear in the 2015 International Conference on Machine Learning.

6. L. Lessard, B. Recht, and A. Packard, "Analysis and design of optimization algorithms via integral quadratic constraints," submitted to the SIAM Journal on Optimization, 2015.

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